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On the Suitability of Hausdorff Dimension as a measure of Texture Complexity

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Abstract. It has been suggested that the Hausdorff dimension of an image texture would be a good measure of the texture's complexity. Binary images can be generated by Büchi automata over a four-letter alphabet. The entropy of the ω-language accepted by such a Büchi automaton has been shown to coincide with the Hausdorff (fractal) dimension of the ω-language if the language is finite-state and closed. Jürgensen and Staiger showed that the (local) entropy of finite-state and closed ω-languages can be effectively computed. Moreover, they postulated that the Hausdorff dimension of the ω-language describing an image texture is a good measure of the texture's complexity or "messiness". If true, then local Hausdorff dimension could be a useful texture feature for image segmentation and classification problems. In this paper we test this claim by examining the results of a survey on the human perception of texture complexity which we recently conducted. Our results indicate that there is virtually no correlation between Hausdorff dimension of texture and the perceived texture complexity.

1 Introduction

Many computer vision and image processing techniques rely on texture features computed from an input image to carry out tasks such as image segmentation and object classification. It has been suggested by Jürgensen and Staiger [1] that local Hausdorff dimension would be a good feature for describing how texture complexity varies over an image. Since it is claimed that Hausdorff dimension would measure the "messiness" or complexity of a texture, an concept with an ill-defined, elusive definition, it would be important to verify that Hausdorff dimension is indeed correlated with human's notion of texture complexity. Unless this correlation exists, the use of Hausdorff dimension as a texture feature would be limited at best.

Thus, we are interested in the relationship between the Hausdorff dimension of ω-languages which describe texture images and the perceived texture complexity of the image described. Culik II and Kari were the first to investigate binary and greyscale image compression using automata [2, 3].

Staiger showed that the entropy of regular ω-languages is computable [4]. It was also shown by Staiger that entropy of a finite-state closed ω-language coincides with the Hausdorff dimension of the language [5]. Jürgensen and Staiger postulated that the Hausdorff dimension of a language would be a good measure of the complexity of a texture generated by a corresponding automaton [6]. Subsequently they defined the local Hausdorff dimension [1] and postulated that constructing a map of local Hausdorff dimension for an image described by a finite-state ω-language would be a good method of illustrating how relative image texture complexity varied over the image [1].

Software was developed software to compute the local Hausdorff dimension of an black and white image in [7].

In this paper we consider whether Hausdorff dimension as a texture feature really measures the the complexity of the texture as perceived by humans. We conducted an experiment to determine whether the Hausdorff dimension coincides with human perception of texture complexity.

2 Quadtrees, Words, Languages and Automata

It is well-known that a binary image can be represented by a finite quadtree. Consider an infinite-resolution binary image. This can be represented by an infinite quadtree Q. The root of Q represents the subsquare coinciding with the entire image. Its children represent the four subquadrants of their parent, and so on. Since Q is infinite, there will be a unique infinitely long path in Q, beginning at the root, for each white point in the image. If we label each edge of the quadtree from the alphabet $\Sigma = \{0, 1, 2, 3\}$, depending on which child of its parent it represents, then the sequence of labels along infinite paths from the root, which correspond to white points in the image, form right-infinite words. The set of all such right-infinite words forms a language $L(Q)$ which is the language of all quadtree addresses of points that are white.

Languages of right-infinite words are called ω-languages. Let X^ω denote the set of all right-infinite words over the alphabet X. As usual, we let X^* denote the set of finite words over X. Note that $L(Q) \subseteq \Sigma^\omega$. Let $w \in \Sigma^*$ and $M \in \Sigma^\omega$. The language

$$w^{[-1]}M = \{\xi \mid \xi \in \Sigma^\omega, w\xi \in M\}$$

is called a state of M. M is said to be finite-state if it has finitely many states.

Let $\mathrm{pref}(M)$ be the set of all finite prefixes of words in M. The closure of M is defined as $C(L) = \{\xi \mid \xi \in \Sigma^\omega, \mathrm{pref}(\{\xi\}) \subseteq \mathrm{pref}(L)\}$. An ω-language is *not* closed if there is a word $\xi \in \Sigma^\omega$ such that $\mathrm{pref}(\{\xi\}) \subseteq \mathrm{pref}(L)$ but $\xi \notin L$.

A Büchi automaton [8], which recognizes an ω-language, is a nondeterministic finite automaton $A = (Q, \Sigma, s, \delta, F)$ with state set Q, alphabet Σ, start state s, transition relation δ and final state set F defined in the usual way, but with a different acceptance condition which we now explain.

Let S be a set. The *infinity set* of a sequence $\sigma = \sigma_0\sigma_1\sigma_2 \ldots \in S^\omega$ is defined as $\mathrm{In}(\sigma) = \{s \in S \mid \exists^\omega n\ \sigma_n = s\}$ where the symbol $\exists^\omega$ denotes the quantifier read as "there exist infinitely many".

A run of an ω-word $\xi = \xi_0\xi_1\xi_2 \ldots \in \Sigma^\omega$ on A is a sequence of states $\sigma = \sigma_0\sigma_1\sigma_2 \ldots$ such that $\sigma_0 = s$ and $(\sigma_i, \xi_{i+1}, \sigma_{i+1}) \in \delta$ for any integer $i > 0$. The ω-word ξ is accepted by A if and only if there is a run σ of ξ on A such that $\mathrm{In}(\sigma) \cap F \neq \emptyset$. The set of ω-languages that can be accepted by a Büchi automaton are called regular ω-languages.

If $L(Q)$ is a regular ω-language, we can construct a Büchi automaton A that accepts $L(Q)$ (see, for example, [7, 3]). The Büchi automaton can then be used to generate the original image at any finite resolution of $2^n \times 2^n$ pixels by determining the acceptance/non-acceptance of every word in Σ^n.

Additional background on the relationships between automata, languages and quadtrees can be found in [9].

Let $G = (V, E)$ be a graph with vertex set V and edge set $E \subseteq V \times V$. For any two vertices $v_1, v_2 \in V$ we write $v_1 \vdash v_2$ if there is an edge from v_1 to v_2, that is, $(v1, v2) \in E$. We denote by $\vdash^*$ the reflexive and transitive closure of $\vdash$. A set of vertices $K \subseteq V$ is a *strongly connected component* of G if, for any two vertices $v_1, v_2 \in K$, we have $v_1 \vdash^* v_2 \vdash^* v_1$ and K is not a subset of any larger such set of vertices. In other words, every vertex in K is reachable from every other vertex in K and K must be as large as possible.

Let A be any finite automaton with state set Q. The underlying graph A_G of A is the graph $G = (V, E)$ such that $V = Q$ and $(q_1, q_2) \in E$ if and only if there is a transition from state q_1 to state q_2 in A. For a set $K \subseteq Q$, the statement that K *is a strongly connected component of* A is equivalent with the statement K *is a strongly connected component of* A_G.

Let $\mathbb{N}_0$ be the set of non-negative integers. For $M \subseteq \Sigma^\omega$, the structure function $s_M(n) : \mathbb{N}_0 \to \mathbb{N}_0$ is defined as

$$s_M(n) = |\mathrm{pref}_n(M)|$$

where $\mathrm{pref}_n(M)$ denotes the number of finite prefixes of length n of words in M. The entropy, H_M, of M is then

$$H_M = \lim_{n\to\infty} \sup \left(\frac{1}{n} \log s_M(n) \right) .$$

The global Hausdorff dimension of M is denoted $\dim M$. If M is finite-state and closed, then H_M coincides with $\dim M$ [5]. In particular, if we assume a base 4 logarithm (which is natural due to our alphabet size) in the above formula for H_M, then

$$\dim M = 2H_M. \tag{1}$$

The global entropy was shown to be effectively computable by Staiger [4].

The local Hausdorff dimension is developed formally by Jürgensen and Staiger in [1]. The main result of their paper is that the computation of the entropy of the states of a finite-state and closed ω-language M is reduced to the computation of the eigenvalues of certain submatrices of the transition matrix of M when the indices of the matrix are arranged such that states in the same strongly connected component of the underlying graph are contiguous. Since Hausdorff dimension coincides with entropy for finite-state and closed ω-languages [5], the local entropy computation yields also local Hausdorff dimension.

Software to compute the local Hausdorff dimension of a finite-state closed ω-language was developed as part of [7].

3 Perception Experiments

We wished to test the postulate that Hausdorff dimension is a good measure of texture complexity. To do so we required image textures with a wide range of Hausdorff dimensions. We created random finite automata by randomly generating transition matrices as follows.

Given a maximum number of states n, chosen at random, we generated an automaton with n states. For each state we randomly determined the number of outgoing transitions $m \in [1, 4]$. Then the destination state of each of these m transitions were determined randomly. First it was randomly determined how many, k, of these m transitions will lead to the same state. Then the destination for these k transitions was chosen randomly. The destinations for the remaining $m - k$ edges were then chosen randomly from among the remaining states so that no destination state was chosen twice. The labels on each of the outgoing transitions were assigned unique labels from 0 to 3 at random. The result of this process was an automaton where each state has between 1 and 4 outgoing transitions and each outgoing transition has a unique label between 0 and 3.

We ensured that the languages accepted by each of the generated automata were closed by designating every state of each automaton as final ($F = Q$ for each automaton).

We desired that the underlying graphs of the generated automata have only a single strongly connected component, that is, where the entire underlying graph of the automata is strongly connected. This ensures that the resulting image produced by the automaton has the same local entropy everywhere. We thus extracted the largest strongly connected component from the underlying graph of each automaton and saved this sub-automaton. In this way we obtained a large number of suitable automata having between 17 and about 20,000 states.

We then calculated the local entropy of each state of each automaton generated in this way using the algorithm from [1] and software developed in [7]. Since the automata were restricted so that the underlying graph of each automaton had only a single strongly connected component, all states in an automaton had the same entropy and the computed entropy value is the also the global Hausdorff dimension. This was taken as the entropy of the texture generated by the automaton.

From these automata we selected a number of automata with entropy in the range $[\frac{1}{2}, 1]$. We did not consider textures with entropy less than $\frac{1}{2}$ because, according to Equation 1, languages with entropy less than $\frac{1}{2}$ will generate objects with dimension less than 1 and thus will not look like textures. We therefore only considered automata which generated images with Hausdorff dimension between 1 and 2.

For those automata selected from the desired entropy range, we generated, for each automaton, a 256 by 256 pixel binary image by determining for each $x \in \Sigma^8$ whether $x \in L$ by running x on the automaton. If $x \in L$, then the pixel addressed by x in the output image was colored white, otherwise it was colored black.

We then designed a survey questionnaire to be distributed to the study participants. We selected two sets of nine images from those that we generated. Images were selected so that their Hausdorff dimensions covered the range $[\frac{1}{2}, 1]$ relatively evenly. Moreover, selected images were also required to have the appearance of a uniform texture. There were six images that were common to both sets. These two sets of images formed the basis for our first two survey questions. For each set of nine images, we presented all nine images on a single page and asked the participants to "examine the following textures, labeled A through I, and order them from most complex to least complex". Participants were not given a definition of "texture complexity".

Our third and final survey question consisted of five pairs of images whose difference in entropy varied from approximately 0.032 to 0.16. We asked the participants to examine the textures and to indicate, for each pair of images, which of the images they believed to be "more complex". The participants could also choose to indicate that they believed that the two images in a pair were approximately the "same complexity".

All surveys were reproduced on the same copy machine to ensure that each participant saw the same images and were not influenced by possible distortions introduced by different reproduction methods.

The surveys were distributed at the University of Saskatchewan and at the University of Western Ontario. There were 101 respondents. Three respondents failed to complete or gave invalid responses to question 1 and four respondents failed to complete or gave invalid responses to question 2. All respondents completed question 3 with valid responses. Data from incomplete surveys were retained for those questions that were answered with valid responses. See Table 1 for a summary of respondents and numbers of data sets retained and discarded.

The set of nine images which formed the first test set are shown in the Appendix.

	Question 1	Question2	Question 3
Total respondents	101	101	101
Incomplete or invalid responses	3	4	0
Number of data sets retained	98	97	101

Table 1. Survey Respondents and Data Sets Retained

4 Results

The left side of Figure 1 shows the average complexity ranking assigned to each image in image set 1 by the survey participants. The survey we distributed asked participants to rank images in order from most complex to least complex, thus lower ranks indicate a perception of higher texture complexity. The x-axis shows the images A through I (as labeled on the survey form) in order from highest texture entropy to lowest entropy. If there is a correlation between texture entropy and perceived complexity we should observe a trend of steadily increasing average rankings from left to right along the x-axis. The right side of Figure 1 shows the standard deviation in the rankings for each image. Each image had a rank standard deviation of around 2 or slightly higher suggesting that there was no significant agreement among participants on the rank of any of the images.

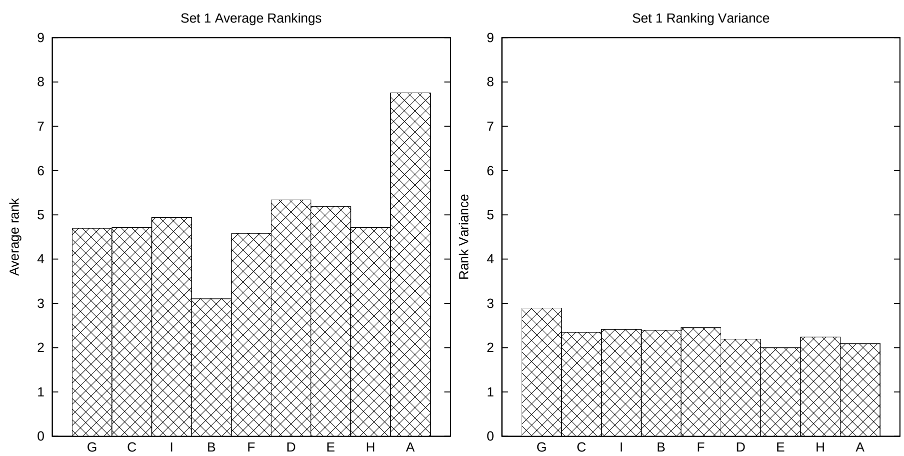

Fig. 1. Average Rankings and Standard Deviation of Rankings for Set 1. Images are shown in order of decreasing entropy.

Figure 2 shows that we obtained results with similar lack of trends for our second set of images. Only three of the images in this data set were different from the images in set 1.

Our purpose in having the overlapping image sets for questions 1 and 2 were to see if perception of complexity was altered if textures were viewed in a different context. We checked to see if there were any respondents who gave opposite relative rankings to the same images in the two data sets. That is, for images A and B common to both sets, we looked for instances where a respondent ranked image A higher than image B in set 1 but ranked B higher than image A in set 2 or *vice versa*. We determined that this occurred with 70 out of 96 respondents (72.9%). There were 184 such instances in total for an average of 1.91 instances per respondent.

To further illustrate the disagreement over rankings among the participants, we have included, as an appendix, a graph for each image in survey questions 1 and 2 showing the percentages of respondents who assigned the image a particular rank. In only a very few cases do we observe any significant agreement by participants on a particular ranking for a particular image.

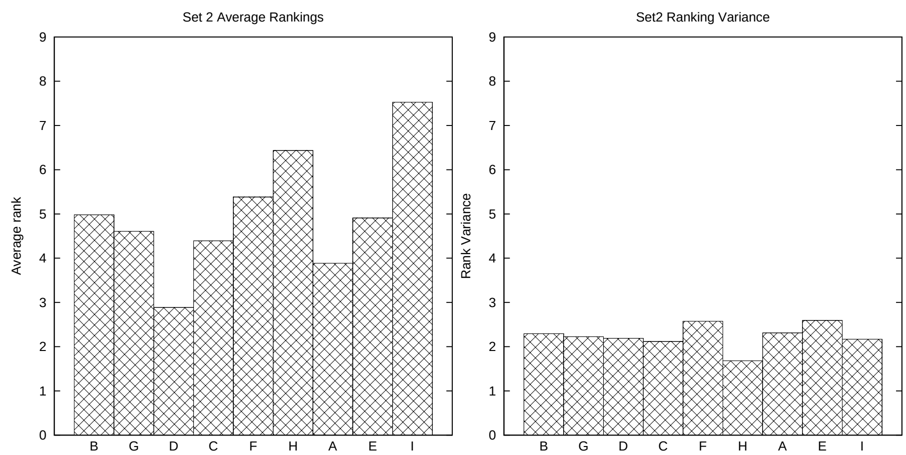

Fig. 2. Average Rankings and Standard Deviation of Rankings for Set 2. Images are shown in order of decreasing entropy.

Table 2 shows the results from question 3 of the survey. Participants were shown five pairs of images and asked to identify for each image pair (A, B) whether A was more complex than B, B was more complex than A, or A and B have the same complexity. In each case image A had a higher entropy than image B. The entropy of each image pair and the difference in entropy between each pair of images is given in Table 3.

In the case of pair 1, the entropy of image B was considerably higher than that of A and 85.1% of participants perceived this. For pair 2, the pair with closest entropy, over 89.1% of the participants believed that the images had the same complexity, or that image B had higher complexity. This is perhaps encouraging evidence of correlation between entropy and perceived complexity given that the entropy of each image texture is so similar. However, in the case of pair 3, whose texture entropy difference is greater than that of pair 2, we see that 65.3% of participants believed that image A was more complex, when in fact image A has lower entropy. There was considerable disagreement over pair 4 with 53.5% of participants perceiving that image A was more complex and 44.6% of participants perceiving that image B was more complex. Lastly, in the case of pair 5, 94.1% of participants perceived that image A was either more complex or of the same complexity as image B with 61.4% perceiving that image A was more complex when in fact image B had higher entropy and a greater entropy difference from its counterpart than pair 4.

Pair Number	A more complex	Same complexity	B more complex
1	11.9	3.0	85.1
2	10.9	38.6	50.5
3	65.3	7.0	27.7
4	53.5	2.0	44.5
5	61.4	32.7	5.9

Table 2. Percentage of respondent's answers for Set 3. In each case the entropy of image A was smaller than that of image B. The first column indicates the pair number. The remaining columns indicate the percentage of respondents who thought that image A was more complex than image B, image A and B were of the same complexity, and image B was more complex than image A, respectively.

Pair Number	Entropy of Image A	Entropy of Image B	Difference
1	0.4034	0.5688	0.1654
2	0.5847	0.6164	0.0317
3	0.6726	0.7299	0.0573
4	0.7900	0.8310	0.0410
5	0.8895	0.9347	0.0452

Table 3. Entropy of image pairs.

5 Conclusions and Open Questions

There appears to be no significant correlation between Hausdorff dimension and perceived texture complexity. There was little agreement among survey participants over the ordering of the image sets from most complex to least complex; moreover, 72.9% of participants reversed the relative rankings of pairs of identical images in set 1 versus set 2 suggesting that perception of complexity may depend on the context in which textures are viewed. This would mean that a measure of texture complexity would have to take into account the surrounding context, which Hausdorff dimension does not.

There was also little agreement over the relative complexity of the five pairs of images presented. In three of the five cases, the majority of respondents perceived that image A was more complex than image B when in fact image A had the lower Hausdorff dimension.

Our conclusion from this study must be that Hausdorff dimension appears to be unsuitable for characterizing image texture complexity.

Since no definition of "complexity" was given to participants, is it possible that some participants interpreted "complexity" to mean complexity in the Kolmogorov sense (the succinctness of the texture description) while others interpreted it to mean "regularity" or "disorder" or even to mean "information content" in the Shannon sense. It is important for a feature that measures texture complexity to be independent of this distinction since many users of image segmentation and classification techniques are unaware of the difference. Thus, if the lack of correlation is due to these different interpretations of "complexity", this would only strengthen the argument that Hausdorff dimension is unsuitable as a measure of texture complexity.

References

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6 Appendix

6.1 Sample Images

The following images made up our first data set of nine images.

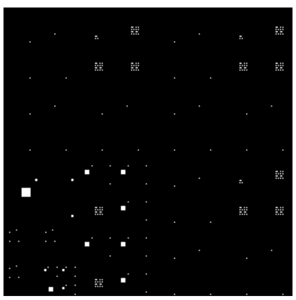


Image A

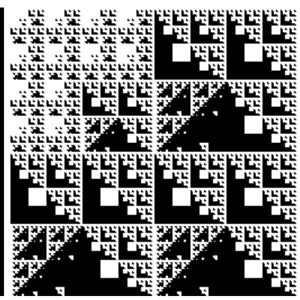


Image B

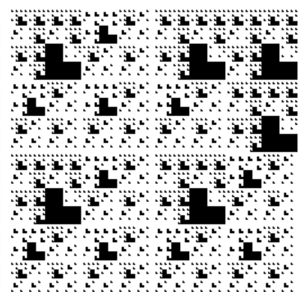


Image C

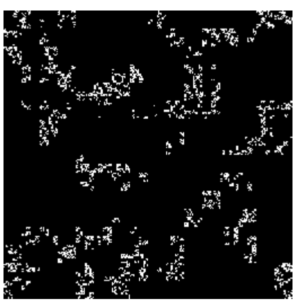


Image D

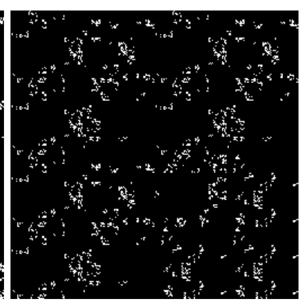


Image E

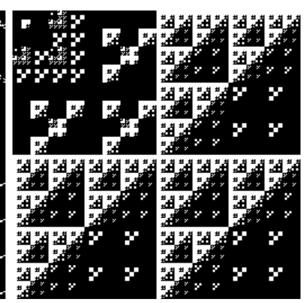


Image F

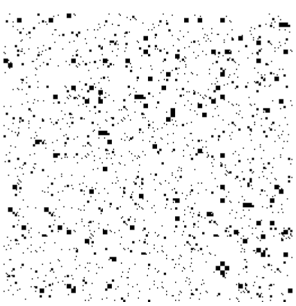


Image G

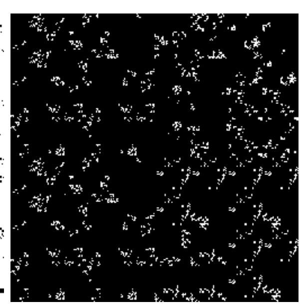


Image H

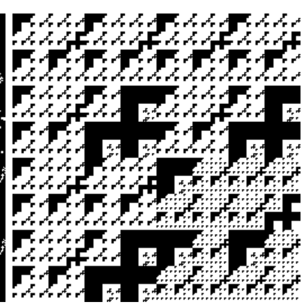
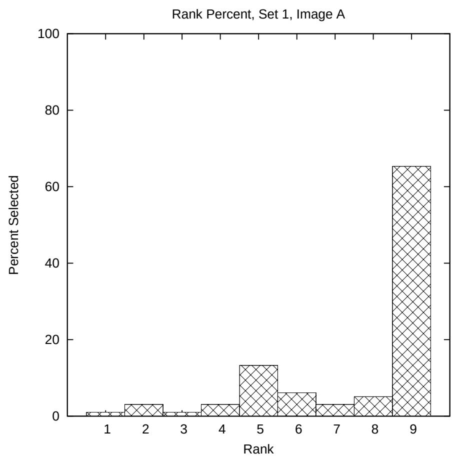
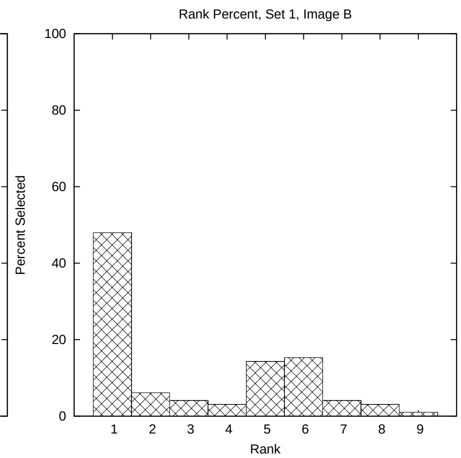
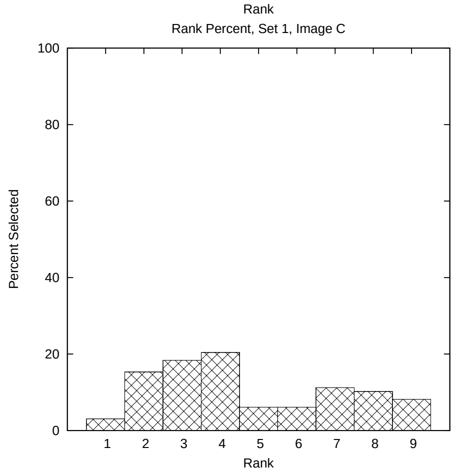


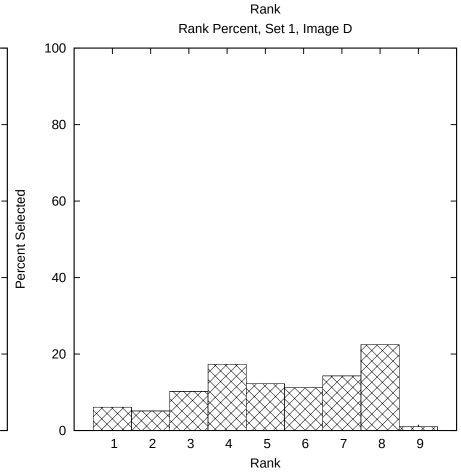
Image I

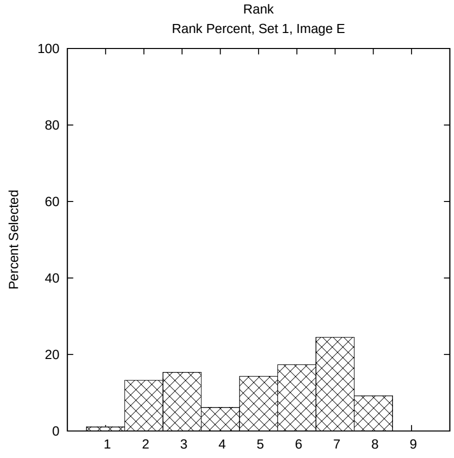
6.2 Rank percentage for Set 1

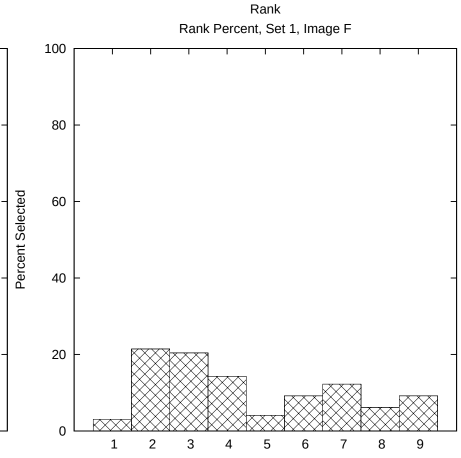


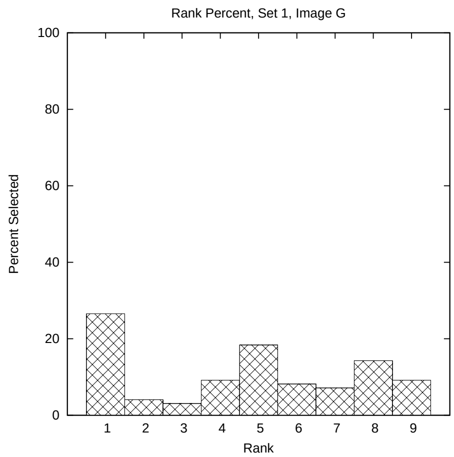











Rank Percent, Set 1, Image G
Percent Selected
Rank
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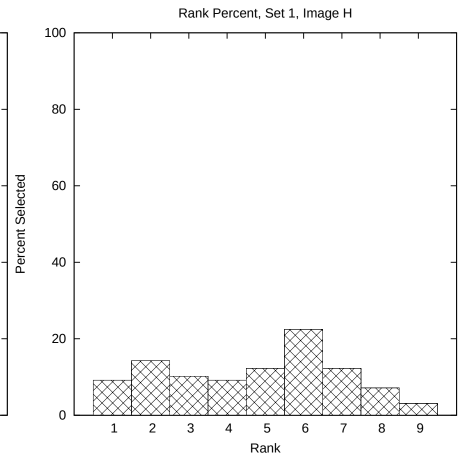
20
40
60
80
100
1
2
3
4
5
6
7
8
9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Percent Selected
Rank
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40
60
80
100
1
2
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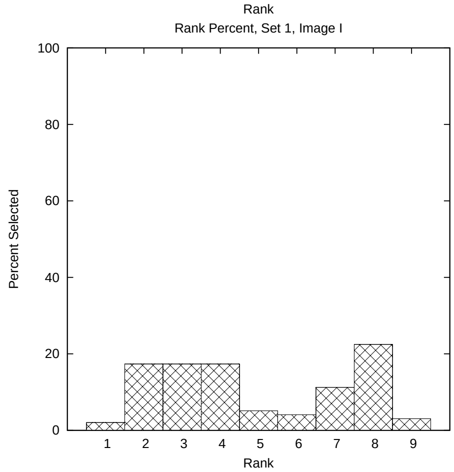
8
9


Rank Percent, Set 1, Image I
Percent Selected
Rank
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40
60
80
100
1
2
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4
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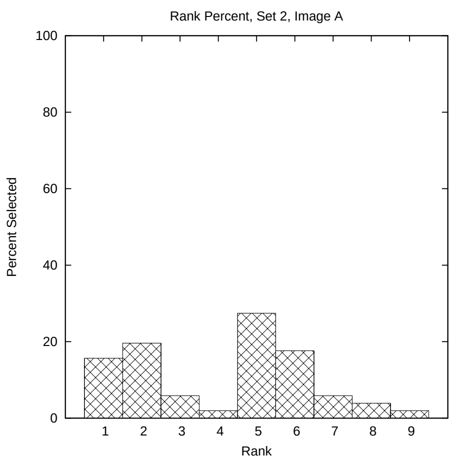
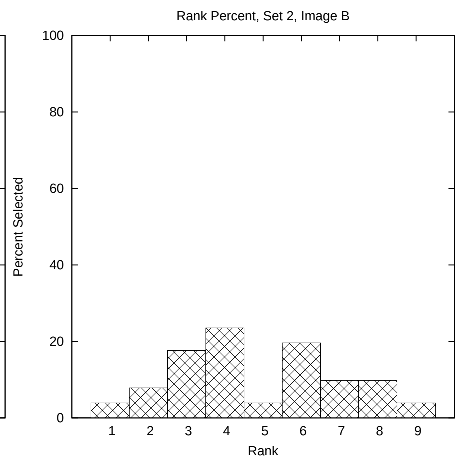
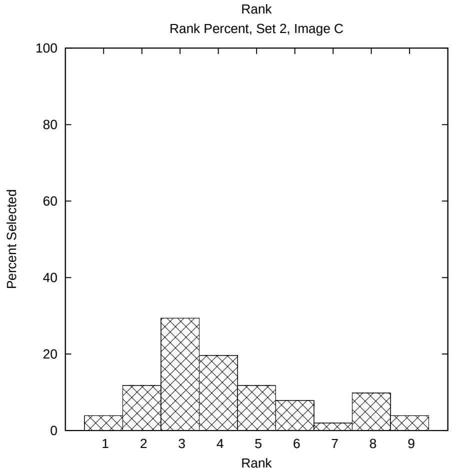
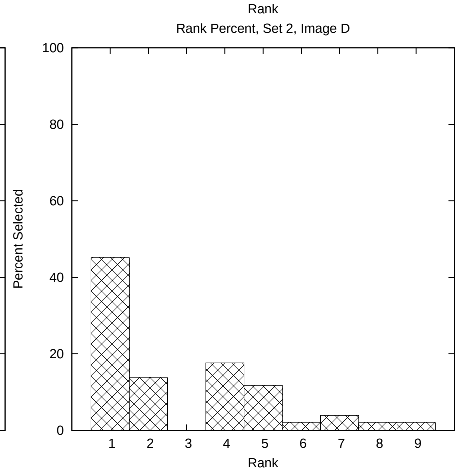
8
9

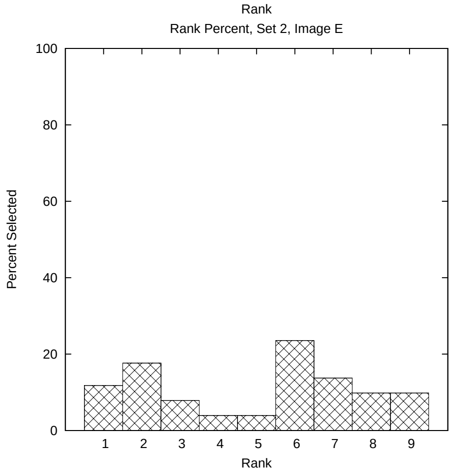
6.3 Rank percentage for Set 2

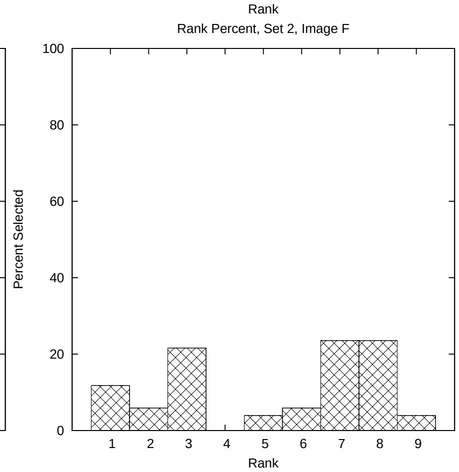


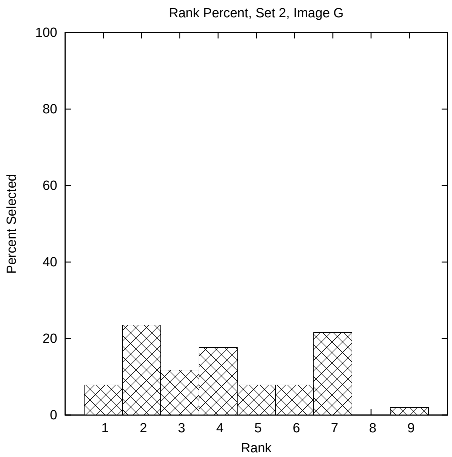








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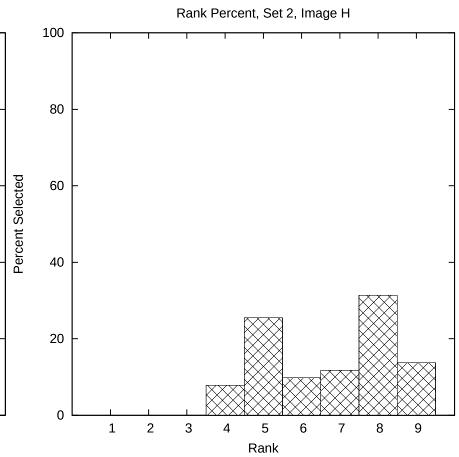
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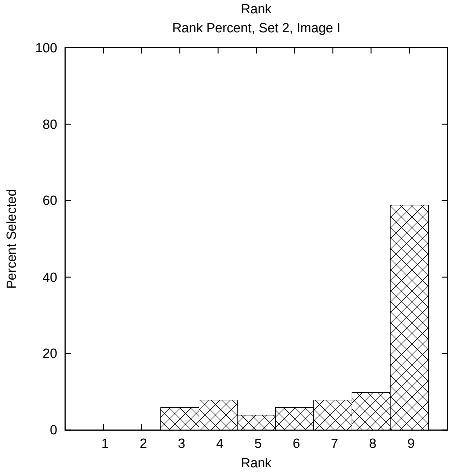
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